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Mass formula for two-dimensional flavor non-singlet mesons

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Abstract. We analytically and numerically investigate the 't Hooft equations, the lowest order mesonic light-front Tamm–Dancoff equations for $SU(N_C)$ and $U(N_C)$ gauge theories, generalized to flavor non-singlet mesons. We find that the wave function can be well approximated by new basis functions and obtain an analytic and an empirical formula for the mass of the lightest bound state. Its value is consistent with the precedent results.

The light front Tamm–Dancoff (LFTD) method [1–3] has been introduced as an alternative tool to lattice gauge theory to investigate relativistic bound states non-perturbatively. In LF coordinates, the physical vacuum is equivalent to the bare vacuum, since all constituents must have non-negative longitudinal momenta defined by $k^+ = (k^0 + k^3)/\sqrt{2}$. Because of this simple structure of the true vacuum, we can avoid the serious problems which appeared in the Tamm–Dancoff (TD) approximation [4] in the equal time frame. Therefore, the TD approximation is commonly used in the context of the LF quantization.

The techniques have been developed [5–10] for solving LFTD equations in several models such as the massive Schwinger model [11], which is the extension of the simplest (1+1)-dimensional QED_2 [12]. Bergknoff [13] first applied the LFTD approximation to the massive Schwinger model. In most of the above references, as they concentrated mainly on taking account of the higher Fock state contributions systematically in the context of the LFTD approximation, they analyzed the LFTD equations assuming that all the masses of quarks are degenerate in order to avoid the complexities of a numerical treatment. Mo and Perry [6] and Harada and coworkers [7] introduced basis functions to treat the massive Schwinger model in the context of the LFTD approximation. One of the present authors [10] generalized their basis functions. But all the basis functions are applicable only in the case where all quark masses are degenerate.

In the real world, as six quarks have their own inherent masses, there are many mesons that consist of a quark and anti-quark with different masses. In this short note, we will attempt to generalize the basis functions so as to treat the masses of mesons consisting of different flavors with different masses. We will neglect the contributions from higher Fock states; then we are led to the generalized 't Hooft–Bergknoff–Eller equation [13,14]

$$\left[M^2 - \frac{m_f^2 - 1}{x} - \frac{m_{f'}^2 - 1}{1-x}\right]\Phi(x) = -\wp\int_0^1 \frac{\Phi(y)}{(y-x)^2}\mathrm{d}y + \alpha\int_0^1 \Phi(y)\mathrm{d}y. \qquad (1)$$

Here, the parameter α specifies the model under consideration, i.e., $\alpha = 0$ for $SU(N_C)$ and $\alpha = 1$ for $U(N_C)$, and $\wp$ stands for the Hadamard finite part. M is the dimensionless meson mass, m_f and $m_{f'}$ are the dimensionless quark mass of flavor f and f'. They are related to the coupling constant g and bare masses $\bar{M}$ and $\bar{m}_f$ as follows:

$$M^2 = \frac{2\pi N_C \bar{M}^2}{(N_C^2 + \alpha - 1)g^2}, \quad m_f^2 = \frac{2\pi N_C \bar{m}_f{}^2}{(N_C^2 + \alpha - 1)g^2}. \qquad (2)$$

One of present authors (O.A.) [10] pointed out that the wave function can be expanded in terms of $(x(1-x))^{\beta_n + j}$ in case $m_{f'} = m_f$. Here, β_n is the $(n+1)$th smallest positive solution of (6) which will be given below. The main interest in the present paper is to extend the basis functions so that we can treat the mesons with $m_f \neq m_{f'}$. One may expect that it is enough to extend the above basis function to $x^{\beta_n + j}(1-x)^{\beta'_n + j}$. As we will see shortly, this is not the case.

At first, according to 't Hooft [14], we put

$$\Phi(x) = x^{\beta}(1-x)^{\beta'}. \qquad (3)$$

The most singular part of the left hand side of (1) at the end point $x = \epsilon$ is given by $-(m_f^2 - 1)\epsilon^{\beta-1}$. The one of the right hand side of (1) is given by

$$-\beta\pi\cot(\pi\beta)\epsilon^{\beta-1}. \qquad (4)$$

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Here we have used

$$\begin{aligned}&\wp\int_0^1 \frac{y^a(1-y)^b}{(y-x)^2}dy\\&= B(a-1,b+1)F(2,1-a-b;2-a;x)\\&\quad -\pi\cot(\pi a)\left\{ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1}\right\}\\&\equiv f_{ab}(x), \qquad (5)\end{aligned}$$

where B is the beta function and F denotes the Gauss hypergeometric function. Thus, we are led to

$$m_f^2 - 1 + \beta\pi\cot(\pi\beta) = 0. \qquad (6)$$

Analogously, we also have at another end the point $x = 1-\epsilon'$,

$${m_f'}^2 - 1 - \beta'\pi\cot(\pi\beta') = 0. \qquad (7)$$

Here, we have used $f_{ab}(x) = f_{ba}(1-x)$. If we assume $\beta \simeq O(m_f)$ for small m_f, we obtain

$$\beta = \frac{\sqrt{3}}{\pi}m_f + \mathcal{O}(m_f^2), \qquad (8)$$

$$\beta' = \frac{\sqrt{3}}{\pi}{m_f}' + \mathcal{O}({m_f'}^2).$$

We multiply both sides of (1) by $\Phi(x)$ and integrate them; we have

$$\begin{aligned}&M^2B(1+2\beta,1+2\beta')\\&= (m_f^2-1)B(2\beta,1+2\beta') + (m_{f'}^2-1)B(1+2\beta,2\beta)\\&\quad -I(\beta,\beta',\beta,\beta') + \alpha B(1+2\beta,1+2\beta'). \qquad (9)\end{aligned}$$

Here,

$$\begin{aligned}&I(a,b,c,d)\\&\equiv \int_{x=0}^1 \wp\int_{y=0}^1 \frac{y^a(1-y)^bx^c(1-x)^d}{(y-x)^2}\mathrm{d}y\mathrm{d}x \qquad (10)\\&= -\pi a\cot(\pi a)B(a+c,b+d)\\&\quad +\pi(a+b)\cot(\pi a)B(1+a+c,b+d)\\&\quad +B(-1+a,1+b)B(1+c,1+d)\\&\qquad \times {}_3F_2(2,1-a-b,1+c;2-a,2+c+d;1).\end{aligned}$$

In the above equation, ${}_3F_2$ denotes the generalized hypergeometric function. Thus, for small m_f and small $m_{f'}$ we are led to

$$M^2 = \alpha + \frac{\pi}{\sqrt{3}}(m_f + m_{f'}) + \mathcal{O}(m_f^2, m_fm_{f'}, {m_{f'}}^2). \qquad (11)$$

As a next step, we consider a higher order correction to (3). The most general form of the wave function is given by

$$\begin{aligned}&\Phi(x) \qquad (12)\\&= \lim_{N\to\infty}\sum_{n_1=0}^{N}\sum_{j_1=0}^{N-n}\sum_{n_2=0}^{N}\sum_{j_2=0}^{N-n} C_{n_1 n_2}^{j_1 j_2} x^{\beta_{n_1}+j_1}(1-x)^{\beta'_{n_2}+j_2},\end{aligned}$$

where β_n and β'_n are the $(n+1)$th smallest positive solution of (6) and (7), respectively. The reason why we cannot introduce a term other than one in (12) will be presented later.

If we substitute (12) into (1), we have, at the end point $x=\epsilon$,

$$\begin{aligned}0 = &-\sum_{n_1=0}^{\infty}(m_f^2 - 1 + \pi\beta_{n_1}\cot\pi\beta_{n_1})\sum_{n_2,j_2} C_{n_1 n_2}^{0\ j_2}\epsilon^{\beta_{n_1}-1} \qquad (13)\\&+\sum_{n_1,J=0}^{\infty}\Bigg[M^2\sum_{k=0}^{J}\sum_{n_2,j_2=0}^{\infty} C_{n_1\ \ n_2}^{J-k j_2}\frac{(-\beta'_{n_2}-j_2)_k}{k!}\\&\quad -(m_f^2-1)\sum_{k=0}^{J+1}\sum_{n_2,j_2=0}^{\infty} C_{n_1\ \ \ \ n_2}^{J+1-k j_2}\frac{(-\beta'_{n_2}-j_2)_k}{k!}\\&\quad -(m_{f'}^2-1)\sum_{k=0}^{J}\sum_{n_2,j_2=0}^{\infty} C_{n_1\ \ n_2}^{J-k j_2}\frac{(-\beta'_{n_2}-j_2+1)_k}{k!}\\&\quad -\sum_{k=0}^{J+1}\sum_{n_2,j_2=0}^{\infty}\pi(\beta_{n_1}+J+1-k)\\&\qquad \times\cot\pi\beta_{n_1} C_{n_1\ \ \ \ n_2}^{J+1-k j_2}\frac{(-\beta'_{n_2}-j_2)_k}{k!}\\&+\sum_{k=0}^{J}\sum_{n_2,j_2}\pi(\beta'_{n_2}+j_2)\\&\quad \times\cot\pi\beta_{n_1} C_{n_1\ \ n_2}^{J-k j_2}\frac{(-\beta'_{n_2}-j_2+1)_k}{k!}\Bigg]\epsilon^{\beta_{n_1}+J}\\&+\sum_{k=0}^{\infty}\sum_{n_1,j_1,n_2,j_2} C_{n_1 n_2}^{j_1 j_2}\Bigg[B(\beta_{n_1}+j_1-1, 1+\beta'_{n_2}+j_2)\\&\quad \times\frac{(2)_k(1-\beta_{n_1}-j_1-\beta'_{n_2}-j_2)_k}{(2-\beta_{n_1}-j_1)_k k!}\\&\quad -\alpha\delta_{k0}B(1+\beta_{n_1}+j_1, 1+\beta'_{n_2}+j_2)\Bigg]\epsilon^k.\end{aligned}$$

Here $(a)_n \equiv \dfrac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. The first line in (14) vanishes automatically, because of the definition of β_n.

Now, it becomes clear that (12) is the most general form. Suppose that we introduce the term like $cx^\gamma(1-x)^{\gamma'}$ with $\gamma \neq \beta_n + j$; then it is required that

$$0 = c(m_f^2 - 1 + \pi\gamma\cot\pi\gamma)\epsilon^{\gamma-1} \qquad (14)$$

holds. Thus, the coefficient c should vanish. Analogously, if $\gamma' \neq \beta'_n + j$ then $c = 0$.

We have a similar equation to (14) at the other end point $x = 1-\epsilon'$ if we truncate (12) to given finite N. The total number of free parameters is $(N+1)^2(N+2)^2/4$. On the other hand, if we require (14) and a similar equation to hold up to order $O(\epsilon^{\beta_{N-1}})$ or $O({\epsilon'}^{\beta'_{N-1}})$, we have

Table 1. Numerical results for the bound state mass M^2 in the $SU(N_C)$ model as a function of the quark masses m_f and $m_{f'}$

		$m_{f'}$					
		0.01	0.02	0.04	0.06	0.08	0.10
m_f	0.01	0.03663	0.05522	0.09293	0.13136	0.17051	0.21036
	0.02	0.05522	0.07398	0.11205	0.15084	0.19034	0.23056
	0.04	0.09293	0.11205	0.15083	0.19033	0.23055	0.27149
	0.06	0.13136	0.15084	0.19033	0.23056	0.27147	0.31312
	0.08	0.17051	0.19034	0.23055	0.27147	0.31317	0.35546
	0.10	0.21036	0.23056	0.27149	0.31312	0.35546	0.39866

Table 2. Numerical results for the bound state mass M^2 in the $U(N_C)$ model as a function of quark masses m_f and $m_{f'}$

		$m_{f'}$					
		0.01	0.02	0.04	0.06	0.08	0.10
m_f	0.01	1.03660	1.05506	1.09233	1.13012	1.16847	1.20738
	0.02	1.05506	1.07387	1.11159	1.14988	1.18867	1.22799
	0.04	1.09233	1.11159	1.15040	1.18954	1.22922	1.26940
	0.06	1.13012	1.14988	1.18954	1.22965	1.27018	1.31122
	0.08	1.16847	1.18867	1.22922	1.27018	1.31154	1.35348
	0.10	1.20738	1.22799	1.26940	1.31122	1.35348	1.39622

$N(N+3)$ independent equations. Thus, we cannot solve the equations in general. We have to reduce the degrees of freedom. We put

$$C^{j_1 j_2}_{n_1 n_2} = \delta_{n_1 0}\delta_{n_2 0}\delta_{j_1 j_2} d_{j_1} + \delta_{j_1 0}\delta_{j_2 0} e_{n_1 n_2}, \tag{15}$$

that is, we assume

$$\Phi(x) = \sum_{j=0}^{N} d_j x^{\beta_0 + j}(1-x)^{\beta'_0 + j} + \sum_{n1,n2=0}^{N} e_{n_1 n_2} x^{\beta_{n_1}}(1-x)^{\beta'_{n_2}}, \tag{16}$$

with $d_{00} = 1$ and $e_{00} = 0$.

For a given m_f and $m_{f'}$, we put $M^2 = M_i^2$. We can then solve (14) for d_j and $e_{n_1 n_2}$ in terms of M_i. We thus obtain the M_i dependent truncated wave function, say, $\Phi(x; M_i)$. We can calculate a new mass eigenvalue M_{i+1} using this wave function by

$$M_{i+1}^2 = \frac{\langle \Phi(M_i)|H|\Phi(M_i)\rangle}{\langle \Phi(M_i)|\Phi(M_i)\rangle}. \tag{17}$$

We can use (11) for M_0^2. For $N \leq 3$, the mass eigenvalue M^2 converges in 5 iterations. For $N = 3$ and $0 < m_f,\ m_{f'} \leq 0.1$, we obtain M^2's by the use of the package Mathematica. The results are summarized in Tables 1 and 2. We can fit them by polynomials:

$$\begin{aligned} &M^2(\alpha = 0, m) \qquad (18) \\ &= 1.8139(m_f + m_{f'}) + 0.892(m_f + m_{f'})^2 + 0.008 m_f m_{f'} \\ &\quad + 0.041(m_f + m_{f'})^3 - 0.18 m_f m_{f'}(m_f + m_{f'}) + \ldots \end{aligned}$$

$$\begin{aligned} &M^2(\alpha = 1, m) \qquad (19) \\ &= 1 + 1.8092(m_f + m_{f'}) + 0.497(m_f + m_{f'})^2 \\ &\quad + 1.564 m_f m_{f'} + 0.95(m_f + m_{f'})^3 \\ &\quad - 4.40 m_f m_{f'}(m_f + m_{f'}) + \ldots \end{aligned}$$

We cannot proceed with this procedure beyond $N = 3$, because we cannot calculate ${}_3F_2(2, a, b; c, d; 1)$ in the desired precision. The results with $m_{f'} = m_f$ are consistent with the previous results in [10].

Finally, we will discuss the so-called "2% discrepancy" problem. In $U(N_C)$, a single flavor model, we expect that the dimensionless meson mass squared can be expanded as follows:

$$M^2 = 1 + b_1 m_f + b_2 m_f^2 + \ldots \tag{20}$$

By the use of the bosonization method, Banks and his coworkers [15] found $b_1 = 2\exp(\gamma_{\mathrm{E}}) = 3.56214\ldots$ Bergknoff [13], however, found $b_1 = 2\pi/\sqrt{3} = 3.62759\ldots$ Two results differ each other by 2%. Our result, (20), is consistent with Bergknoff's result rather than Banks et al.'s. We may expect that $b_1 = 2\pi/\sqrt{3}$ is the minimum value in the context of 't Hooft–Bergknoff–Eller equation and the higher Fock sector should be included to solve the problem.

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